A Simulation Preorder for Koopman-like Lifted Control Systems

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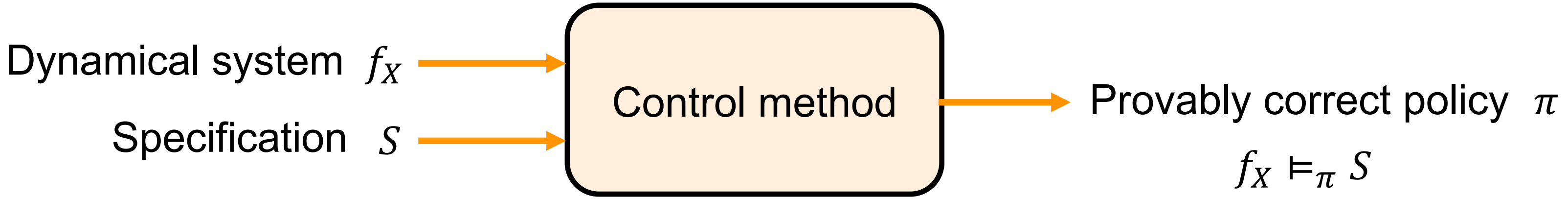
Motivation

Modern control problems are often safety critical

→ Need for provably correct control policies



Challenging for nonlinear systems

Abstraction based techniques [1,2,3]

1. Define a notion of simulation between systems $f_X \preccurlyeq f_Y$
2. Prove that $f_X \preccurlyeq f_Y$ and $f_Y \vDash_\pi S$ implies $f_X \vDash_\pi S$

[1] Baier, C., & Katoen, J. P. (2008). Principles of model checking.
[2] Tabuada, P. (2009). Verification and control of hybrid systems: a symbolic approach.
[3] Reissig, G., Weber, A., & Rungger, M. (2016). Feedback refinement relations for the synthesis of symbolic controllers.

Koopman-inspired simulation

Key idea: Approximate a nonlinear system by a linear one using a lifting function

Given

- a nonlinear systems $x(t+1) = f_X(x(t), u(t))$
- a lifting function $\psi: \mathbb{R}^{n_X} \to \mathbb{R}^{n_Y}: x \mapsto \psi(x) = \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$

consider $y = \psi(x)$ and find

$$y(t+1) = Ay(t) + Bu(t) \qquad \text{s.t.} \qquad \psi\big(f_X(x,u)\big) \approx A\psi(x) + Bu$$

- Most works do not provide formal guarantees [1,2]
- Simulation-like relations for autonomous systems in [3,4]
- Control systems are considered in our previous work [5]

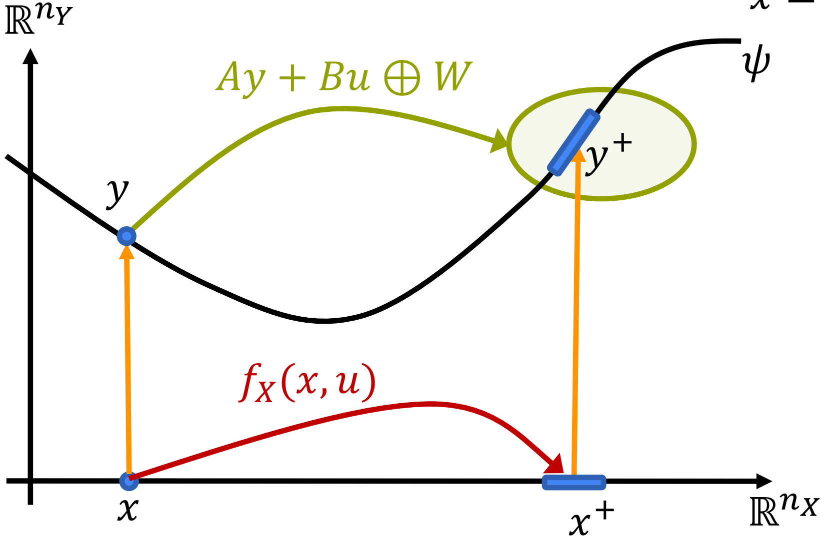
[1] Korda, M., & Mezić, I. (2018). Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control.
[2] Proctor, J. L., Brunton, S. L., & Kutz, J. N. (2018). Generalizing Koopman theory to allow for inputs and control.
[3] Sankaranarayanan, S. (2016). Change-of-bases abstractions for non-linear hybrid systems.
[4] Wang, Z., Jungers, R. M., & Ong, C. J. (2023). Computation of invariant sets via immersion for discrete-time nonlinear systems.
[5] Balim, H., Aspeel, A., Liu, Z., & Ozay, N. (2023). Koopman-inspired implicit backward reachable sets for unknown nonlinear systems.

Previous work [5]

Polyhedra

$$\psi(f_X(x,u)) \subseteq A\psi(x) + Bu \oplus W \qquad \psi(x) = \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$$

$$x^+ \in f_X(x,u) \quad \underset{x = [I \;\; 0]y}{\overset{y = \psi(x)}{\rightleftarrows}} \quad y^+ \in Ay + Bu \oplus W$$



Allows to abstract a nonlinear system into an affine lifted system

Question: How to choose ψ ?

→ Need to compare lifted systems

[5] Balim, H., Aspeel, A., Liu, Z., & Ozay, N. (2023). Koopman-inspired implicit backward reachable sets for unknown nonlinear systems.

Outline

1. Lifted systems

2. Preordering lifted systems

3. Computational aspects (and challenges)

4. Future works and conclusion

Lifted systems

A **lifted system** is a tuple $LS_Y = (X, U, \psi_Y, f_Y, g_Y)$, where

- $X \subseteq \mathbb{R}^{n_X}$ a set of outputs
- U a set of inputs
- $\psi_Y: \mathbb{R}^{n_X} \to \mathbb{R}^{n_Y}$ a lifting function
- $f_Y: \mathbb{R}^{n_Y} \times U \rightrightarrows \mathbb{R}^{n_Y}$ a set-valued dynamics
- $g_Y: \mathbb{R}^{n_Y} \to \mathbb{R}^{n_X}$ an output map such that $g_Y\big(\psi_Y(x)\big) = x$

$$\psi_Y(x) = \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$$

$$f_Y(y, u) = A_Y y + B_Y u \oplus W_Y$$

$$g_Y(y) = [I \quad 0]y$$

A **solution** of LS_Y is a tuple $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}) \in X^{[0,T[} \times U^{[0,T[} \times (\mathbb{R}^{n_Y})^{[0,T[}$ s.t.

- $y(0) = \psi_Y\big(x(0)\big)$
- $y(t+1) \in f_Y\big(y(t), u(t)\big)$
- $x(t) = g_Y(y(t))$

Lifted systems

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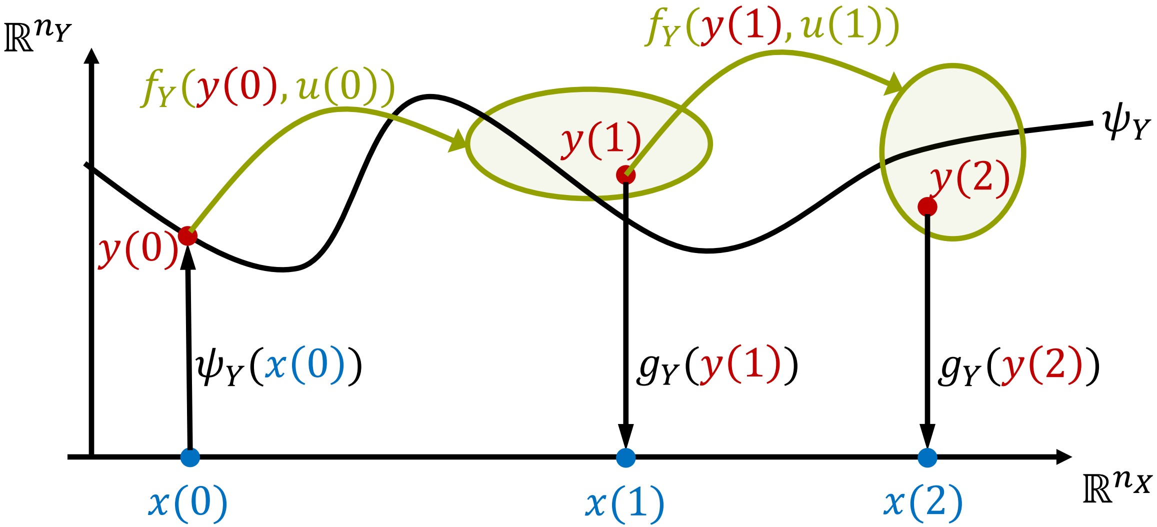


Important classes of lifted systems

A **lifted system** is a tuple $LS_Y = (X, U, \psi_Y, f_Y, g_Y)$

- **Unlifted** (i.e., “classical”) systems

$$x^+ \in f_X(x, u)$$

are lifted systems with $n_Y = n_X$ and $\psi_Y = g_Y = id$. Concrete systems of interest are unlifted

- **Affine** lifted systems

Polyhedra

$$\begin{aligned} y^+ &\in Ay + Bu \oplus W \\ x &= Cy \end{aligned}$$

for which linear control methods can be used

- **Piecewise affine** lifted systems

Behavior and Specifications

The **closed-loop behavior** of LS_Y under a policy π is the set:

$$\boldsymbol{B}_\pi[LS_Y] = \{\ (\boldsymbol{x}, \boldsymbol{u}) \mid\ \exists \boldsymbol{y} \text{ s.t. } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}) \text{ is a max.}^* \text{ solution } \&\ u(t) = \pi\big(x(0), \dots, x(t)\big)\ \}$$

A **specification** is a set of finite or infinite sequences of (x, u) pairs: $S \subseteq (X \times U)^\infty$

e.g., safety or linear temporal logic

The specification S is **satisfied** by the lifted system LS_Y under the policy π if $\boldsymbol{B}_\pi[LS_Y] \subseteq S$.
This is written $LS_Y \vDash_\pi S$

* A solution is **maximal** if it is either:

i. infinite, i.e., $T = \infty$
ii. blocking, i.e., $f_Y\big(y(T), u(T)\big) = \emptyset$
iii. possibly leaving the output set, i.e., $g_Y\left(f_Y\big(y(T), u(T)\big)\right) \nsubseteq X$

Simulation relation for lifted systems

LS_Y is **simulated** by LS_Z (denoted $LS_Y \preccurlyeq LS_Z$) if there exists a set-valued map $\rho: \mathbb{R}^{n_Z} \rightrightarrows \mathbb{R}^{n_Y}$ such that

- $\forall x \in X$: $\psi_Y(x) \in \rho(\psi_Z(x))$ *(Relation between liftings)*
- $\forall (z,u) \in \mathbb{R}^{n_Z} \times U$: $f_Y(\rho(z),u) \subseteq \rho\big(f_Z(z,u)\big)$ *(between dynamics)**
- $\forall z \in \mathbb{R}^{n_Z}$: $g_Y\big(\rho(z)\big) \subseteq \{g_Z(z)\}$ *(between outputs)*
- a technical condition to handle blockingness

* Similar to alternating simulation.

Simulation implies behavioral inclusion

THEOREM

Given two lifted systems LS_Y and LS_Z and a policy π, if LS_Y is simulated by LS_Z, then the closed-loop behavior of LS_Y under π is included in the closed-loop behavior of LS_Z under π, i.e.,

$$LS_Y \preccurlyeq LS_Z \quad \Longrightarrow \quad \boldsymbol{B}_\pi[LS_Y] \subseteq \boldsymbol{B}_\pi[LS_Z]$$

COROLLARY

$$LS_Y \preccurlyeq LS_Z \vDash_\pi S \quad \Longrightarrow \quad LS_Y \vDash_\pi S$$

Some special cases of $LS_Y \preccurlyeq LS_Z$

If LS_Y is unlifted (i.e., “classical”) and

- LS_Z is affine
  → **Koopman over-approximation** [1]

- LS_Z is affine and both systems are autonomous (i.e., $U = \{0\}$)
  → **Approximate immersion** [2]

- LS_Z is piecewise affine and unlifted
  → **Hybridization** [3]

[1] Balim, H., Aspeel, A., Liu, Z., & Ozay, N. (2023). Koopman-inspired implicit backward reachable sets for unknown nonlinear systems.
[2] Wang, Z., Jungers, R. M., & Ong, C. J. (2023). Computation of invariant sets via immersion for discrete-time nonlinear systems.
[3] Girard, A., & Martin, S. (2011). Synthesis for constrained nonlinear systems using hybridization and robust controllers on simplices.

In practice…

Given an unlifted (i.e., “classical”) system LS_X

Step 1: Pick K lifting functions $\psi_1, \dots, \psi_K$

Step 2: For each, compute an affine lifted system LS_k s.t. $LS_X \preccurlyeq LS_k$

Step 3: If $LS_i \preccurlyeq LS_j$, delete LS_j

Step 4: Use one of the remaining lifted system to synthesize a policy

Computing affine lifted systems (Step 2)

Given

- an unlifted (i.e., “classical”) system LS_X
- a lifting function ψ

find an affine lifted system LS_Y s.t. $LS_X \preccurlyeq LS_Y$

Find A, B and W s.t.

$$\forall (x,u) \in X \times U: \quad \psi\big(f_X(x,u)\big) \subseteq A\psi(x) + Bu \oplus W \qquad (*)$$

with $W = \{w \mid Hw \le h\}$

$$\boxed{\min_{A,B,h} \|h\|_1} \quad \text{s.t.} \quad (*)$$

Minimize nondeterminism

ASSUMPTIONS

- X and U are polyhedra
- $f_X(x,u)$ is single valued and polynomial
- ψ is polynomial

Can be handled using Sum-of-Squares optimization

Illustration with backward reachable sets (BRS)

DEFINITION (informal)

The **BRS** of $x^+ \in f(x, u)$ is the set of points from which there exists a control sequence leading to a given target set, while staying in a safety set.

Illustration with backward reachable sets (BRS)

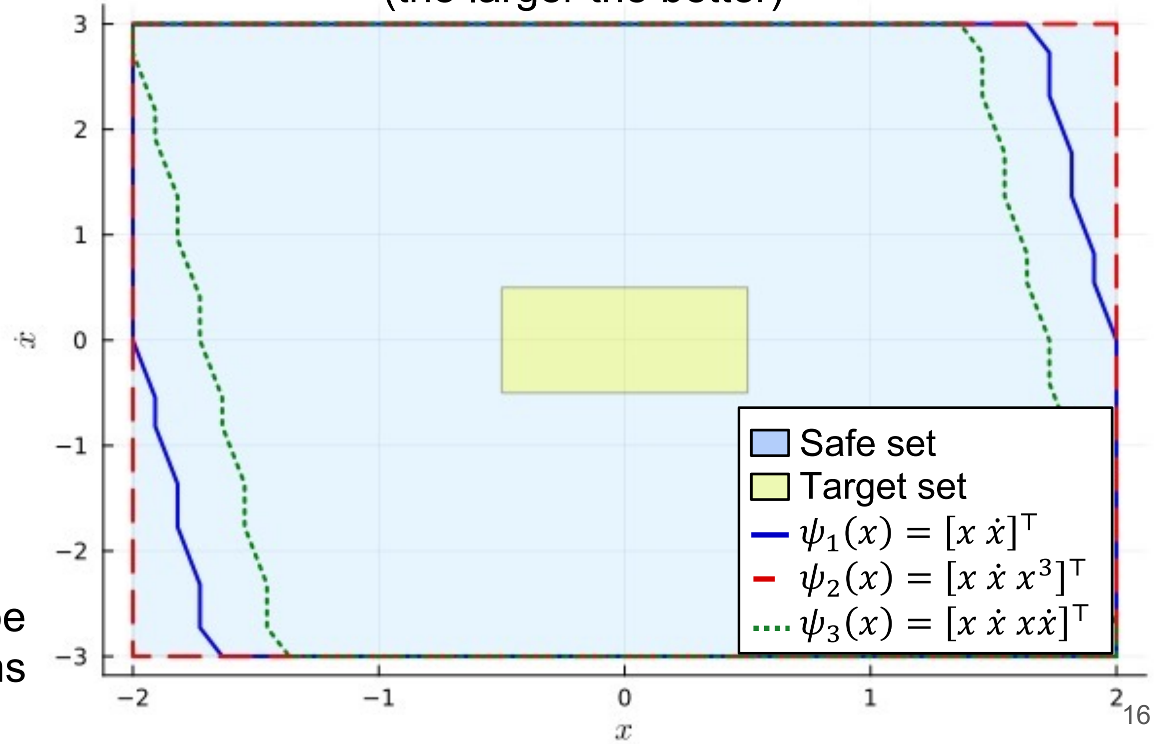
$$\ddot{x} = 2x - 2x^3 - 0.5\,\dot{x} + u$$

$$\psi_1(x) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\psi_2(x) = \begin{bmatrix} x \\ \dot{x} \\ x^3 \end{bmatrix}$$

$$\psi_3(x) = \begin{bmatrix} x \\ \dot{x} \\ x\dot{x} \end{bmatrix}$$

(the larger the better)



Inner-approximations of BRS can be computed using affine lifted systems

Comparing affine lifted systems (Step 3)

Given two affine lifted systems LS_Y and LS_Z, verifying if $LS_Y \preccurlyeq LS_Z$ is a feasibility problem:

Find $\boxed{\rho: \mathbb{R}^{n_Z} \rightrightarrows \mathbb{R}^{n_Y}}$ s.t.

lives in an infinite dimensional space

- $\forall x \in X$: $\quad \psi_Y(x) \in \rho\big(\psi_Z(x)\big)$
- $\forall (z,u) \in \mathbb{R}^{n_Z} \times U$: $\quad A_Y \rho(z) + B_Y u \oplus W_Y \subseteq \rho(A_Z z + B_Z u \oplus W_Z)$
- $\forall z \in \mathbb{R}^{n_Z}$: $\quad \begin{bmatrix} I_{n_Y} & 0 \end{bmatrix} \rho(z) \subseteq \left\{ \begin{bmatrix} I_{n_Z} & 0 \end{bmatrix} z \right\}$

infinite number of constraints

We derived **finite-dimensional** sufficient conditions by assuming $\rho(z) = Rz \oplus W$

Unfortunately, it is **too conservative** to be useful in practice…

Take home message

- Preorder between lifted systems
- Lifted systems can be used as abstractions
- A first (theoretical) step towards a method to select the lifting function ψ

Limitations and future works

- Need to improve computational methods
  → Find finite dimensional parameterizations for $\rho(z)$ that reduces conservatism
- Generalize to continuous-time and hybrid systems

Thank you!

Any questions?

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