A low-Rank Approach to Minimize Sensor-to-Actuator Communication in Finite-Horizon Output Feedback

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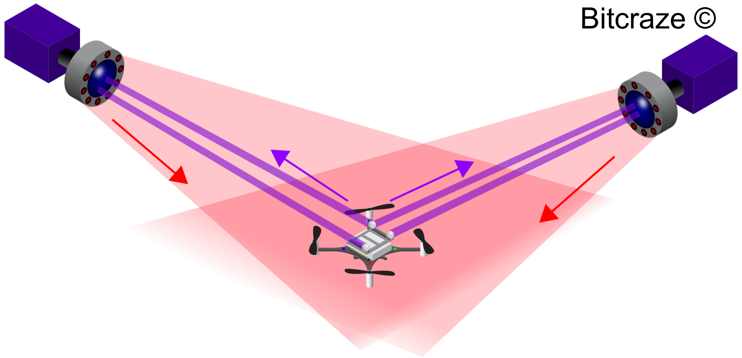
Motivations

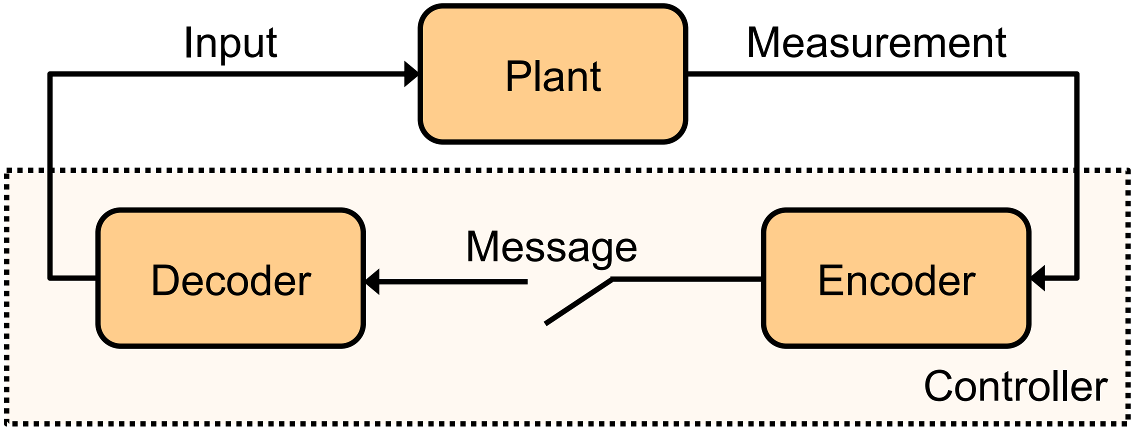
Modern control systems are composed of distributed components interacting via a communication network

→ Limited communications



Motion capture system in a drone arena

Minimize the number of communications from sensor to actuator



Related works

Self-/Event triggered control

- Heemels, W. P., Johansson, K. H., & Tabuada, P. An introduction to event-triggered and self-triggered control. *CDC* 2012.

  → Most works focus on triggering sensors or actuators (not communication)

Approximate a given controller to reduce communication

- Braksmayer, M., & Mirkin, L. Redesign of stabilizing discrete-time controllers to accommodate intermittent sampling. *IFAC-PapersOnLine*, 2017.

  → Do not minimize communications

Problem statement

- Linear discrete time system

$$x_{t+1} = Ax_t + Bu_t + w_t$$
$$y_t = Cx_t + v_t$$

finite horizon $t = 1, \dots, T$

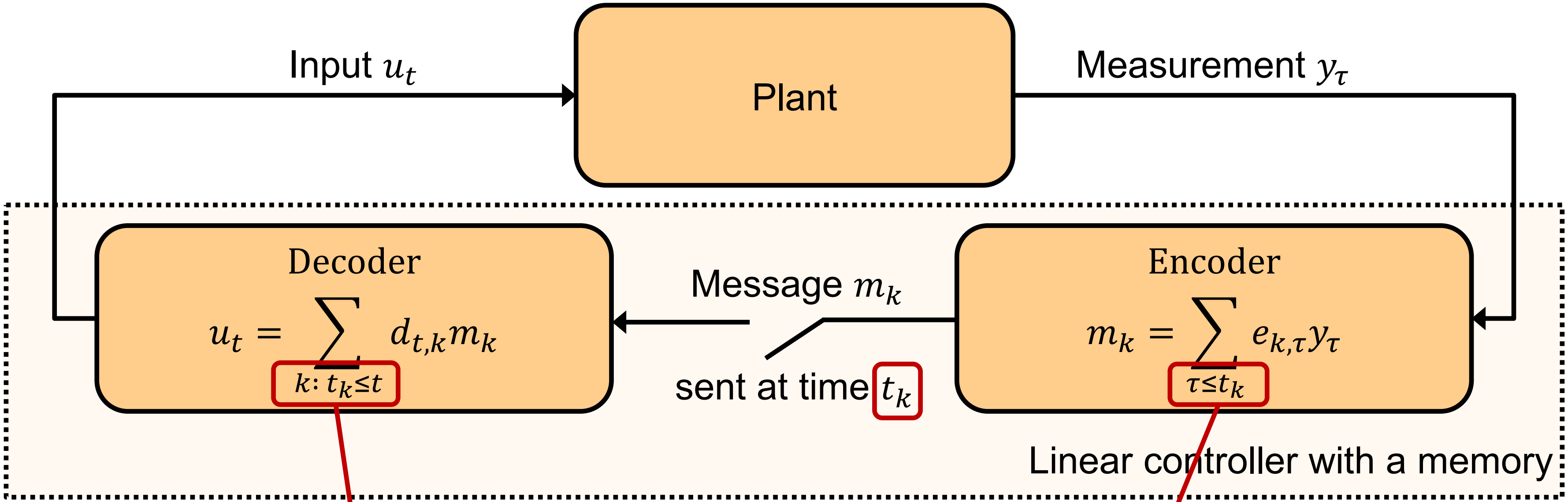
- Safety constraint:

  For all $w_t \in W$, $v_t \in V$, and $x_1 \in X_{\text{init}}$, it holds that $u_t \in U$ and $x_t \in X$ for all t

- Minimize the number r of messages (will be precised soon)

To simplify notation: SISO

Controller structure



Causality:
(i) m_k uses only measurements received before t_k
(ii) u_t uses only messages received before t

Controller structure: matrix formulation

Decoder: $u_t = \sum_{k:\, t_k \le t} d_{t,k} m_k$ ← Message m_k sent at time t_k ← Encoder: $m_k = \sum_{\tau \le t_k} e_{k,\tau} y_\tau$

$\boldsymbol{y} = [y_1 \quad y_2 \quad \cdots \quad y_T]^\top$ $\quad \boldsymbol{m} = [m_1 \quad m_2 \quad \cdots \quad m_r]^\top$ $\quad \boldsymbol{u} = [u_1 \quad u_2 \quad \cdots \quad u_T]^\top$

Decoder $\mathbf{D} \in \mathbb{R}^{T \times r}$: $\boldsymbol{u} = \mathbf{D}\boldsymbol{m}$ ← Message m_k sent at time t_k ← Encoder $\mathbf{E} \in \mathbb{R}^{r \times T}$: $\boldsymbol{m} = \mathbf{E}\boldsymbol{y}$

Causality:
There exists $t_1, \dots, t_r$ such that $\quad \mathbf{E}_{k,\tau} = \mathbf{D}_{t,k} = 0 \quad$ for $t < t_k < \tau$

→ Optimize over $\mathbf{D}$ and $\mathbf{E}$ under causality constraint

Problem statement: **MINIMUM MESSAGES PROBLEM**

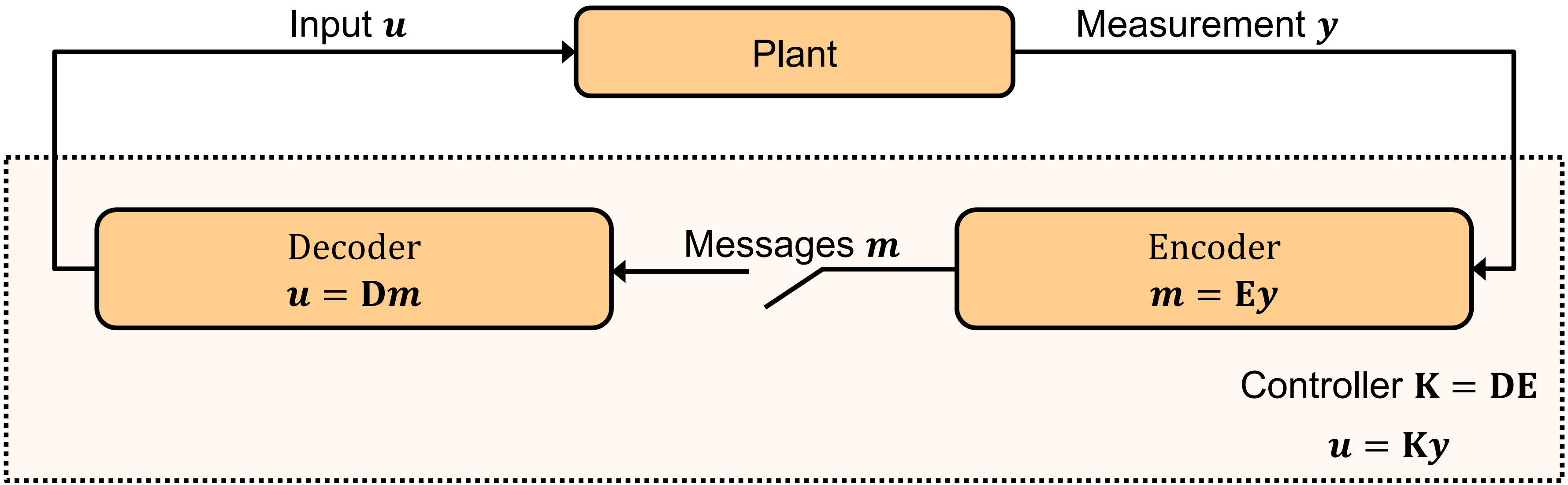
Find the minimum r such that there exists

- $\mathbf{E} \in \mathbb{R}^{r \times T}$
- $\mathbf{D} \in \mathbb{R}^{T \times r}$
- $1 \leq t_1 \leq \cdots \leq t_r \leq T$

satisfying

1. Controller is causal: for all $k = 1, \dots, r$: $\mathbf{E}_{k,\tau} = \mathbf{D}_{t,k} = 0$ for $t < t_k < \tau$
2. Controller closes the loop: $\boldsymbol{m} = \mathbf{E}\boldsymbol{y}, \quad \boldsymbol{u} = \mathbf{D}\boldsymbol{m}$
3. Linear dynamics: $x_{t+1} = Ax_t + Bu_t + w_t, \quad y_t = Cx_t + v_t$
4. Safety constraints: For all $w_t \in W$, $v_t \in V$, and $x_1 \in X_{\text{init}}$: $u_t \in U$ and $x_t \in X$ for all t

Problem statement



Optimizing over t_k, $\mathbf{D}$ and $\mathbf{E}$ is hard… (combinatorial, bilinear)
→ Uses a 2 steps approach

1. Optimize over $\mathbf{K}$
2. Factorize $\mathbf{K}$ into $(\mathbf{D}, \mathbf{E})$ and recover the t_k

Causal factorization

DEFINITION

Given a triangular matrix $\mathbf{K} \in \mathbb{R}^{T \times T}$, a **causal factorization** of $\mathbf{K}$ is a pair $(\mathbf{D}, \mathbf{E})$ s.t.

1. $\mathbf{E} \in \mathbb{R}^{r \times T}$ *(r rows)*
2. $\mathbf{D} \in \mathbb{R}^{T \times r}$ *(r columns)*
3. $\mathbf{K} = \mathbf{DE}$ *(factorization)*
4. There exits $1 \leq t_1 \leq \cdots \leq t_r \leq T$ s.t. for all $k = 1, \ldots, r$ *(causality)*

$$\mathbf{E}_{k,\tau} = \mathbf{D}_{t,k} = 0 \qquad \text{for } t < t_k < \tau$$

The number of messages r is the **band** of the factorization.

Observation: Since $\mathbf{K} = \mathbf{DE}$, $r \geq \operatorname{rank} \mathbf{K}$

Question: Is there always a factorization of band $r = \operatorname{rank} \mathbf{K}$?

Rank minimization

Question: Is there always a factorization of band $r = \operatorname{rank} \mathbf{K}$? **Yes!**

THEOREM 1

1. Any triangular matrix $\mathbf{K} \in \mathbb{R}^{T \times T}$ admits a causal factorization with band $r = \operatorname{rank} \mathbf{K}$
2. There is a polynomial time algorithm to compute such a causal factorization

COROLLARY

An optimal solution to the **MINIMUM MESSAGES PROBLEM** can be obtained by

1. Solving

$$\min_{\mathbf{K}} \operatorname{rank} \mathbf{K} \quad \text{s.t.} \quad (\text{Triang.}), \quad (\text{System Dynamics}), \quad (\text{Safety}), \quad \boldsymbol{u} = \mathbf{K}\boldsymbol{y}$$

2. Computing a causal factorization $(\mathbf{D}, \mathbf{E})$ of $\mathbf{K}$ with band equal to $\operatorname{rank} \mathbf{K}$

Rank minimization

COROLLARY

An optimal solution to the **MINIMUM MESSAGES PROBLEM** can be obtained by

1. Solving

$$\min_{\mathbf{K}} \operatorname{rank} \mathbf{K} \qquad \text{s.t.} \qquad (\text{Triang.}), \qquad (\text{System Dynamics}), \qquad (\text{Safety}), \qquad \boldsymbol{u} = \mathbf{K}\boldsymbol{y}$$

2. Computing a causal factorization $(\mathbf{D}, \mathbf{E})$ of $\mathbf{K}$ with band equal to $\operatorname{rank} \mathbf{K}$

- No combinatorial constraint due to t_k anymore!
- Safety constraint is non-convex in $\mathbf{K}$

→ Use Youla-parameterization [1] or System Level Synthesis (SLS) [2]

[1] Skaf, Boyd. Design of affine controllers via convex optimization. *TAC*, 2010.
[2] Anderson, Doyle, Low, Matni. System level synthesis. *Annual Reviews in Control*, 2019

System level synthesis

THEOREM (adapted from [1])

All system responses $\{\mathbf{\Phi}_{\mathbf{xx}}, \mathbf{\Phi}_{\mathbf{xy}}, \mathbf{\Phi}_{\mathbf{ux}}, \mathbf{\Phi}_{\mathbf{uy}}\}$

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{\mathbf{xx}} & \mathbf{\Phi}_{\mathbf{xy}} \\ \mathbf{\Phi}_{\mathbf{ux}} & \mathbf{\Phi}_{\mathbf{uy}} \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}$$

achievable by a triangular $\boldsymbol{K}$ (and only those) are triangular and satisfy

$$\mathbf{\Phi}_{\mathbf{xx}}, \mathbf{\Phi}_{\mathbf{xy}}, \mathbf{\Phi}_{\mathbf{ux}}, \mathbf{\Phi}_{\mathbf{uy}} \in \mathrm{AffineSpace}$$

In addition, $\mathbf{K} = \mathbf{\Phi}_{\mathbf{uy}} - \mathbf{\Phi}_{\mathbf{ux}} \mathbf{\Phi}_{\mathbf{xx}}^{-1} \mathbf{\Phi}_{\mathbf{xy}}$

The safety constraint is convex in $\{\mathbf{\Phi}_{\mathbf{xx}}, \mathbf{\Phi}_{\mathbf{xy}}, \mathbf{\Phi}_{\mathbf{ux}}, \mathbf{\Phi}_{\mathbf{uy}}\}$ (using Farkas' lemma)

Question: How to express $\mathrm{rank}\, \mathbf{K}$ in terms of $\{\mathbf{\Phi}_{\mathbf{xx}}, \mathbf{\Phi}_{\mathbf{xy}}, \mathbf{\Phi}_{\mathbf{ux}}, \mathbf{\Phi}_{\mathbf{uy}}\}$

[1] Hassaan, Yong. System-level recurrent state estimators for affine systems subject to data losses Modeled by automata. *CDC*, 2022

System level synthesis

Question: How to express $\mathrm{rank}\,\mathbf{K}$ in terms of $\{\mathbf{\Phi}_{\mathbf{xx}}, \mathbf{\Phi}_{\mathbf{xy}}, \mathbf{\Phi}_{\mathbf{ux}}, \mathbf{\Phi}_{\mathbf{uy}}\}$

THEOREM 2

If $\{\mathbf{\Phi}_{\mathbf{xx}}, \mathbf{\Phi}_{\mathbf{xy}}, \mathbf{\Phi}_{\mathbf{ux}}, \mathbf{\Phi}_{\mathbf{uy}}\}$ are triangular and satisfy

$$\mathbf{\Phi}_{\mathbf{xx}}, \mathbf{\Phi}_{\mathbf{xy}}, \mathbf{\Phi}_{\mathbf{ux}}, \mathbf{\Phi}_{\mathbf{uy}} \in \mathrm{AffineSpace}$$

and $\mathbf{K} = \mathbf{\Phi}_{\mathbf{uy}} - \mathbf{\Phi}_{\mathbf{ux}}\mathbf{\Phi}_{\mathbf{xx}}^{-1}\mathbf{\Phi}_{\mathbf{xy}}$, then

$$\mathrm{rank}\,\mathbf{K} = \mathrm{rank}\,\mathbf{\Phi}_{\mathbf{uy}}$$

The big picture…

COROLLARY

An optimal solution to the **MINIMUM MESSAGES PROBLEM** can be obtained by

1. Solving

$$\min_{\mathbf{\Phi_{xx}},\mathbf{\Phi_{xy}},\mathbf{\Phi_{ux}},\mathbf{\Phi_{uy}}} \ \operatorname{rank} \mathbf{\Phi_{uy}} \quad \text{s.t.} \quad (\text{Triang.}), \qquad \mathbf{\Phi_{xx}}, \mathbf{\Phi_{xy}}, \mathbf{\Phi_{ux}}, \mathbf{\Phi_{uy}} \in \text{AffineSpace}, \qquad (\text{Safety})$$

2. Computing $\mathbf{K} = \mathbf{\Phi_{uy}} - \mathbf{\Phi_{ux}}\mathbf{\Phi_{xx}^{-1}}\mathbf{\Phi_{xy}}$
3. Computing a causal factorization $(\mathbf{D}, \mathbf{E})$ of $\mathbf{K}$ with band equal to $\operatorname{rank} \mathbf{K}$.

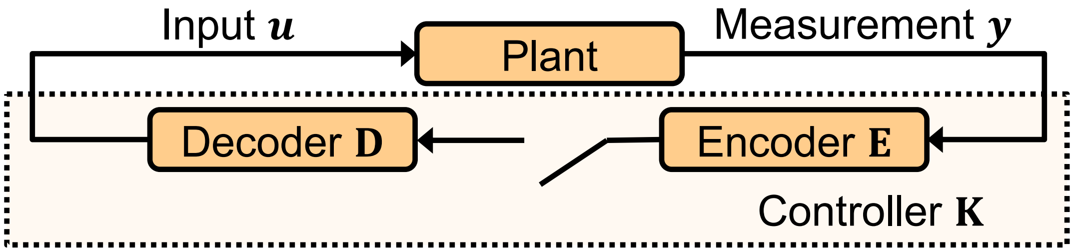
- Constraints are convex (even linear)!
- Rank minimization is NP-hard

→ Use convex relaxation (nuclear norm) [1]: $\operatorname{rank} \mathbf{\Phi_{uy}} \rightarrow \left\|\mathbf{\Phi_{uy}}\right\|_*$

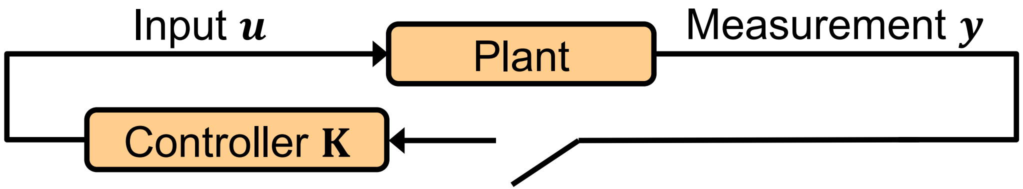
[1] Mohan, Fazel. Reweighted nuclear norm minimization with application to system identification. *ACC,* 2010

Benchmarks

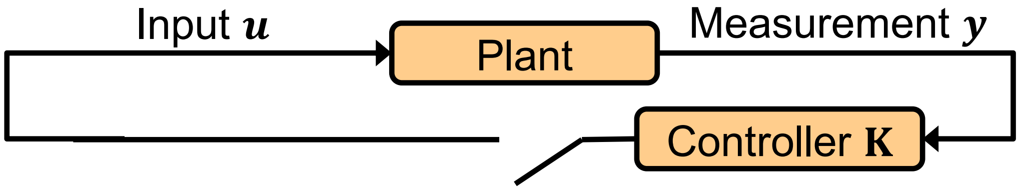
Minimize messages



Minimize sensor usage



Minimize actuator usage



Numerical experiments

2D double integrator (drone model)

$$\ddot{p}_x = u_x$$
$$\ddot{p}_y = u_y$$

Discretized ($T = 20$ time steps)

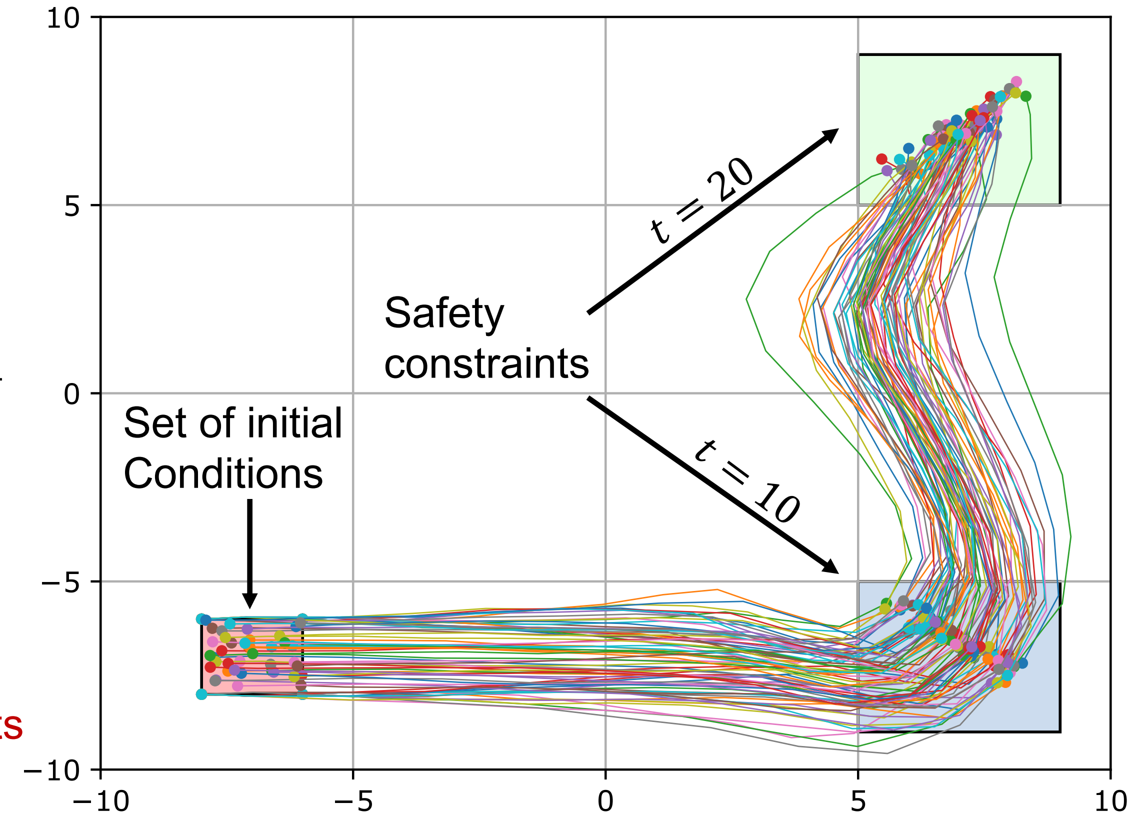
Measure the position only $[p_x \quad p_y]^\top$

Add bounded process noise w_t and measurement noise v_t

Min. messages: 13 messages

Min. sensor usage: 16 measurements

Min. actuator usage: 26 actuations



Take home message

- To minimize sensor-to-actuator messages → split the controller in encoder/decoder
- System level synthesis can be used to have convex constraints

Future works

- Approximate causal factorization $\mathbf{K} \approx \mathbf{DE}$
- Infinite horizon (e.g., using FIR)

Thank you!

Any questions?

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