Koopman-inspired Implicit Backward Reachable Sets for Unknown Nonlinear Systems

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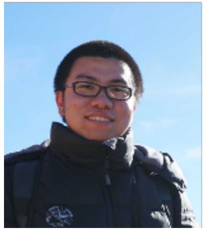

Prof. Necmiye Ozay

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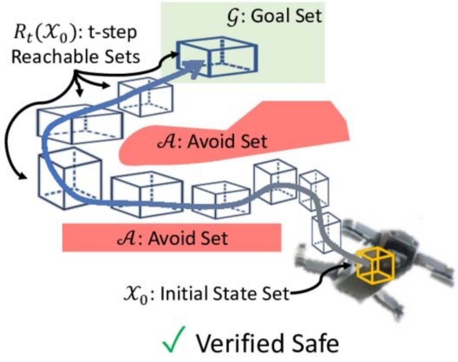
CLEVR-AI

Motivation

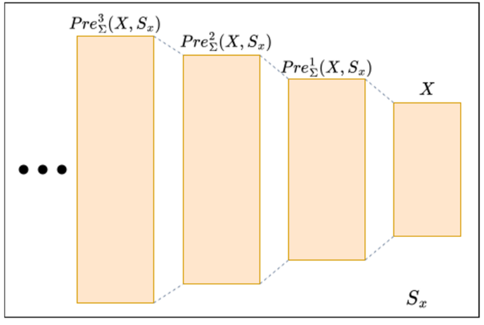
- **Data-driven:** Solutions that harness the power of data.
- **Robustness:** Approaches that are robust to modelling errors and uncertainties.
- **Verification:** Control algorithms that are provably safe.



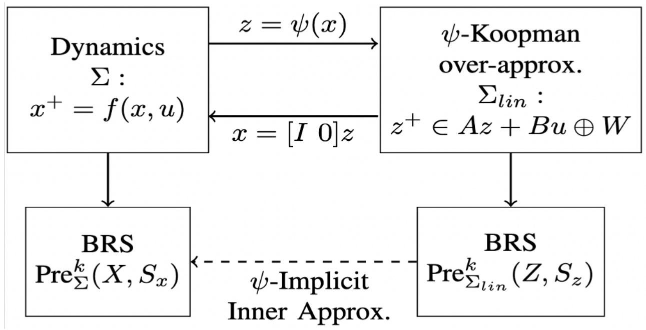
Figures are adapted from [1,2]

[1] M. Everett, G. Habibi, C. Sun, and J. P. How, "Reachability analysis of neural feedback loops"
[2] J. Kabzan, L. Hewing, A. Liniger, and M. N. Zeilinger, "Learning-based model predictive control for autonomous racing"

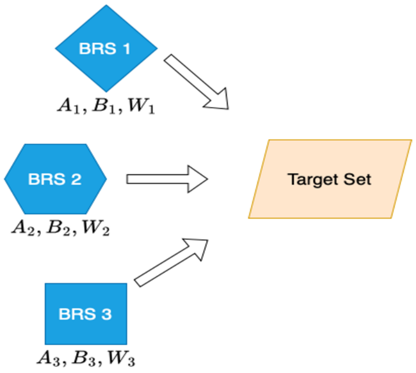
Outline



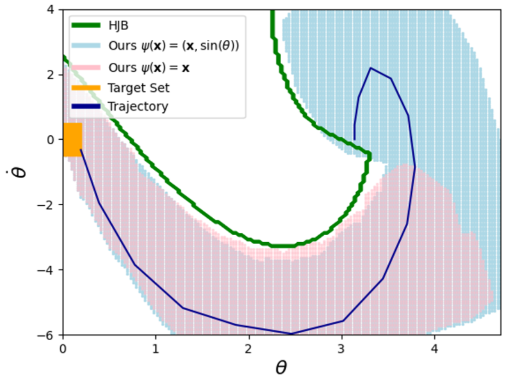
Backward Reachable Sets



Koopman Over Approximations (KOA)



Local KOA



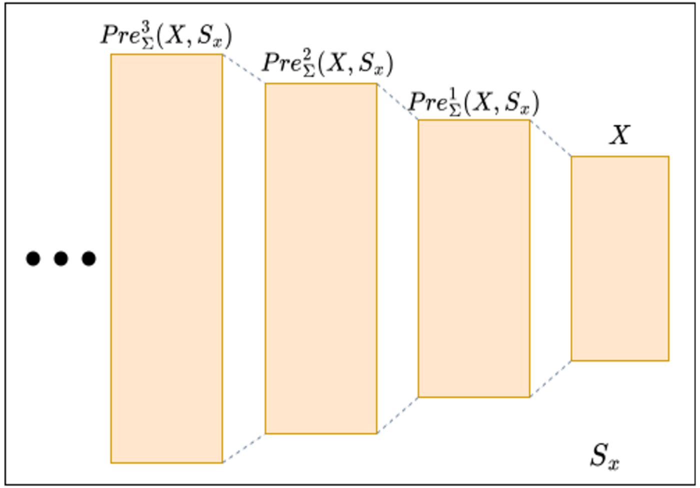
Experiments

Backward reachable set (BRS)

- We consider discrete time nonlinear systems of the form: $x^+ = f(x, u)$

- Given a target set, compute the backward reachable set.
  - Recursively compute longer horizon

- Having BRS simplifies controller design significantly



Problem statement

- For nonlinear systems, computing the BRS is intractable
  - BRS can also be nonconvex
  - Common set representations cannot be used

- Goal: Compute **inner-approximations** of the BRS for unknown nonlinear systems.

- Existing methods posses weaknesses

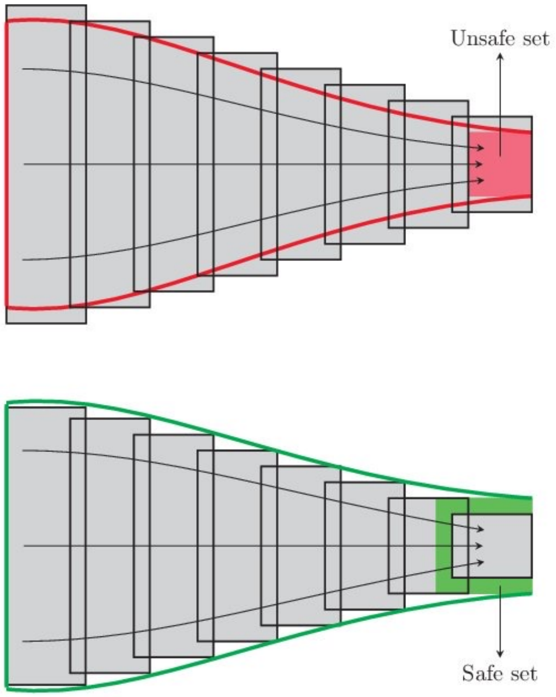


Figure from [3]

[3] D. Han, A. Rizaldi, A. El-Guindy, and M. Althoff, “On enlarging backward reachable sets via zonotopic set membership”

Related Work

- Hamilton-Jacobi-Bellman (HJB) equation solvers [4, 5]:
    - Cannot scale to high-dimensional problems
    - Most approaches do not guarantee inner-approximation
- Set-based approaches [6, 7]:
    - Limited to linear systems
    - Linearization-based approaches exist but they have large approximation error
- Koopman theory: Approximate nonlinear systems by linear ones using a lifting function [8]

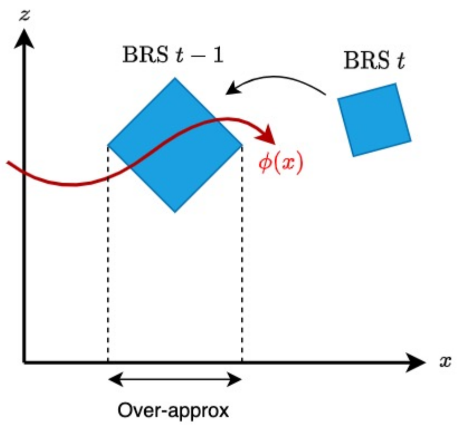
$$\psi : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z} : x \mapsto \psi(x) = \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$$

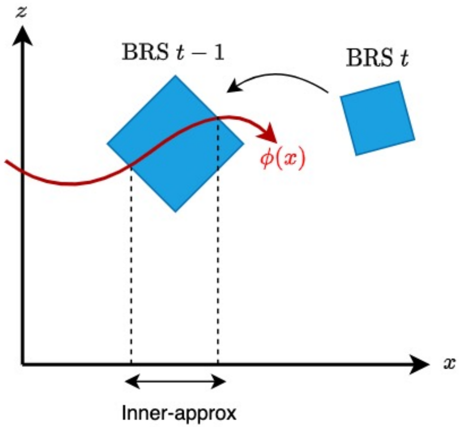
$$\psi(f(x,u)) \approx A\psi(x) + Bu$$

[4] S. Bansal, M. Chen, S. Herbert, and C. J. Tomlin, "Hamilton-jacobi reachability: A brief overview and recent advances"
[5] I. Mitchell, A. Bayen, and C. Tomlin, "A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games"
[6] M. Althoff, G. Frehse, and A. Girard, "Set propagation techniques for reachability analysis"
[7] L. Yang, H. Zhang, J.-B. Jeannin, and N. Ozay, "Efficient backward reachability using the Minkowski difference of constrained zonotopes"
[8] S. L. Brunton, M. Budisic, E. Kaiser, and J. N. Kutz, "Modern Koopman theory for dynamical systems"

Koopman over-approximation

- Compute BRS for linear system robust to modelling error
- The computed sets cannot be directly used
- For inner-approximation intersection with manifold is needed
- Intersection is still non-convex



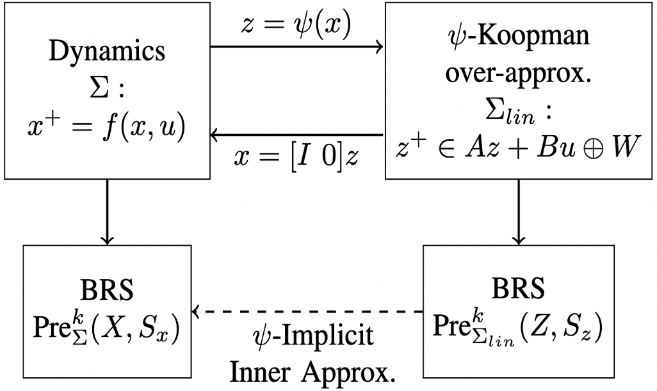


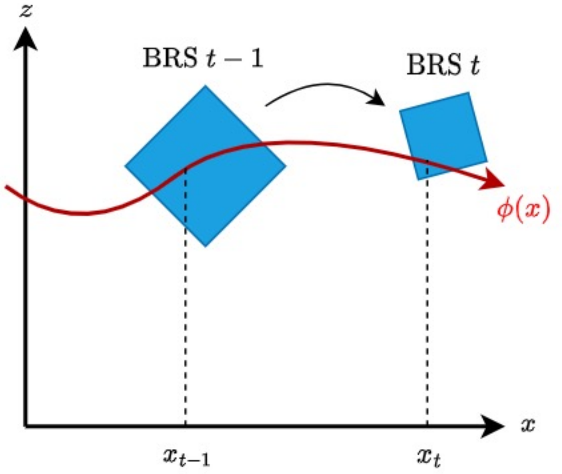
Main Result

Theorem If

- Σ_{lin} is a ψ-Koopman over-approximation of Σ
- S_z is a ψ-implicit inner-approximation of S_x
- Z is a ψ-implicit inner-approximation of X

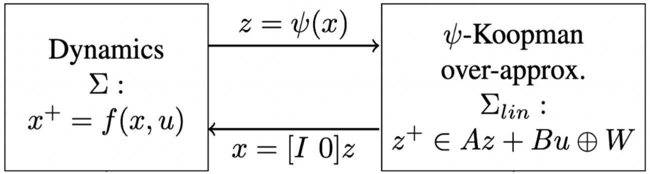
then $Pre^t_{\Sigma_{lin}}(Z, S_z)$ is a ψ-implicit inner-approximation of $Pre^t_{\Sigma}(X, S_x)$ for all t

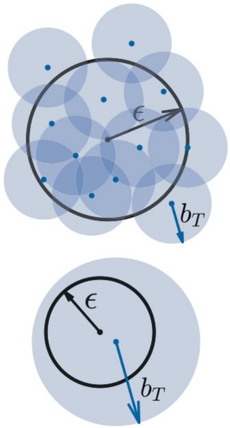


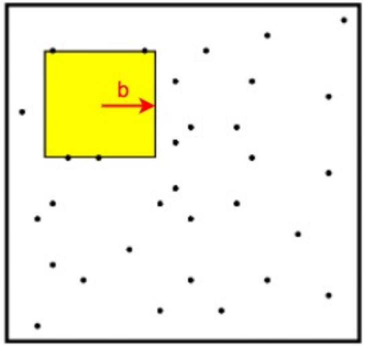


Data-driven Koopman over-approximations

- The system matrices are estimated from data
- The error bound is computed using:
  - Maximum train error
  - Error functions Lipschitz constant
- Lipschitz constant is estimated via Extreme Value Theory [9]





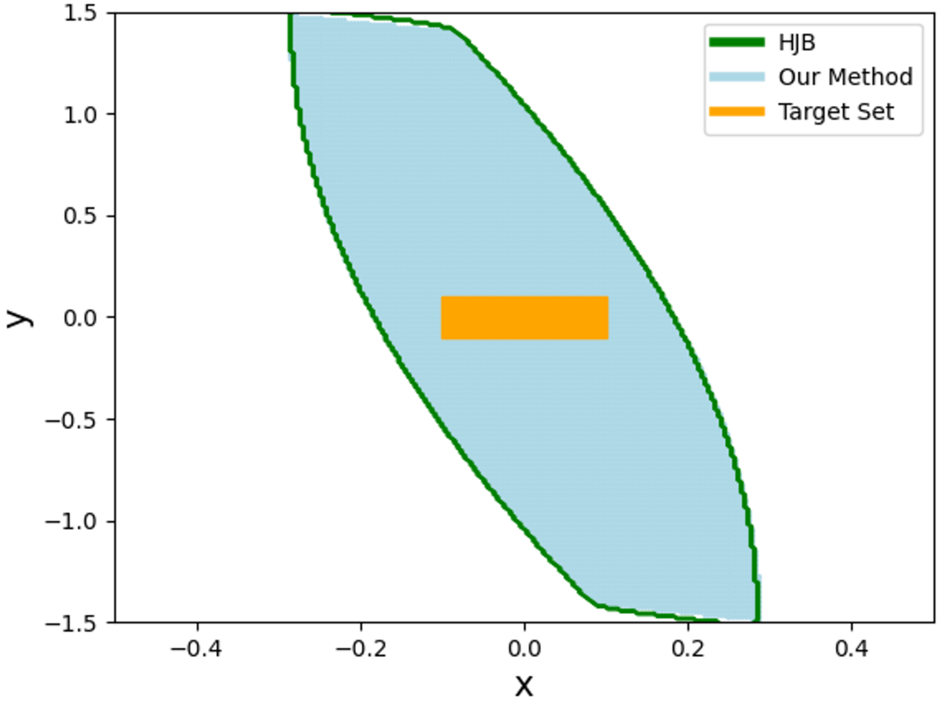


Figures adapted from [10, 11]

[9] G. Wood, and B. Zhang, “Estimation of the Lipschitz constant of a function”
[10] C. Knuth, G. Chou, N. Ozay, and D. Berenson “Probabilistic guarantees on safety and reachability via Lipschitz constants”
[11] S.M. LaValle, “Planning algorithms”

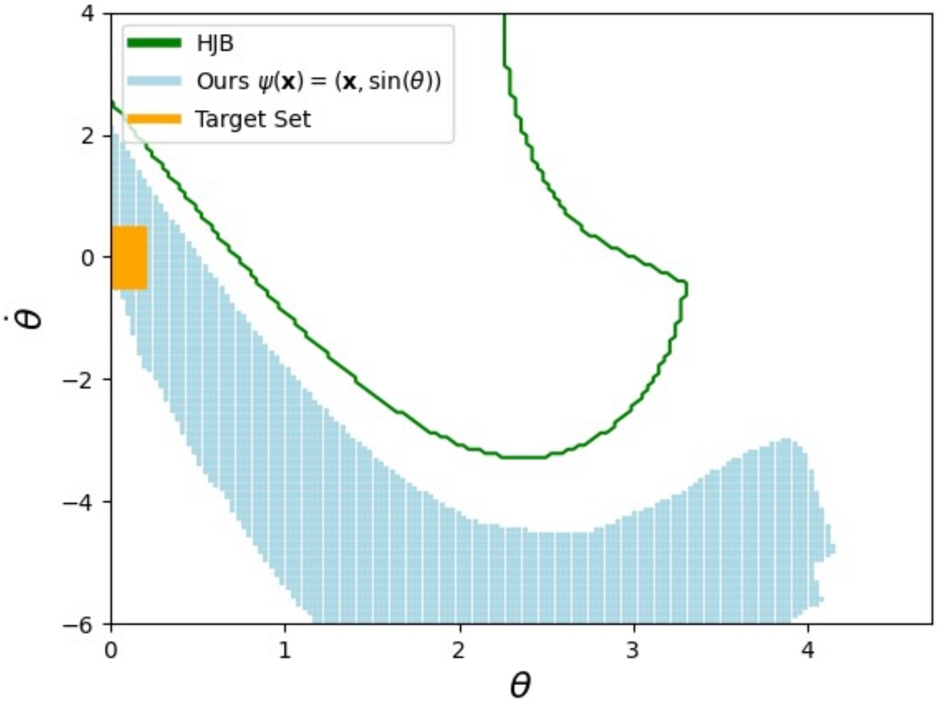
Example: Forced duffing oscillator

$$\dot{\mathbf{x}} = \begin{bmatrix} y \\ 2x - 2x^3 - 0.5y + u \end{bmatrix}$$



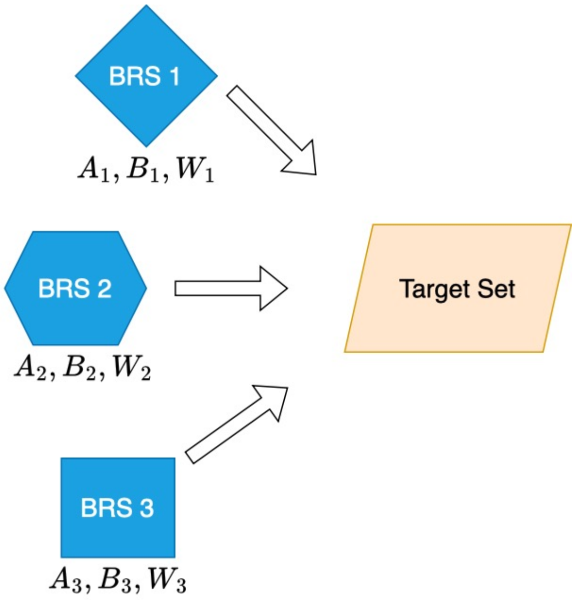
Example: Inverted Pendulum

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{\theta} \\ \frac{3g}{2l}\sin(\theta) + \frac{3}{ml^2}u \end{bmatrix}$$



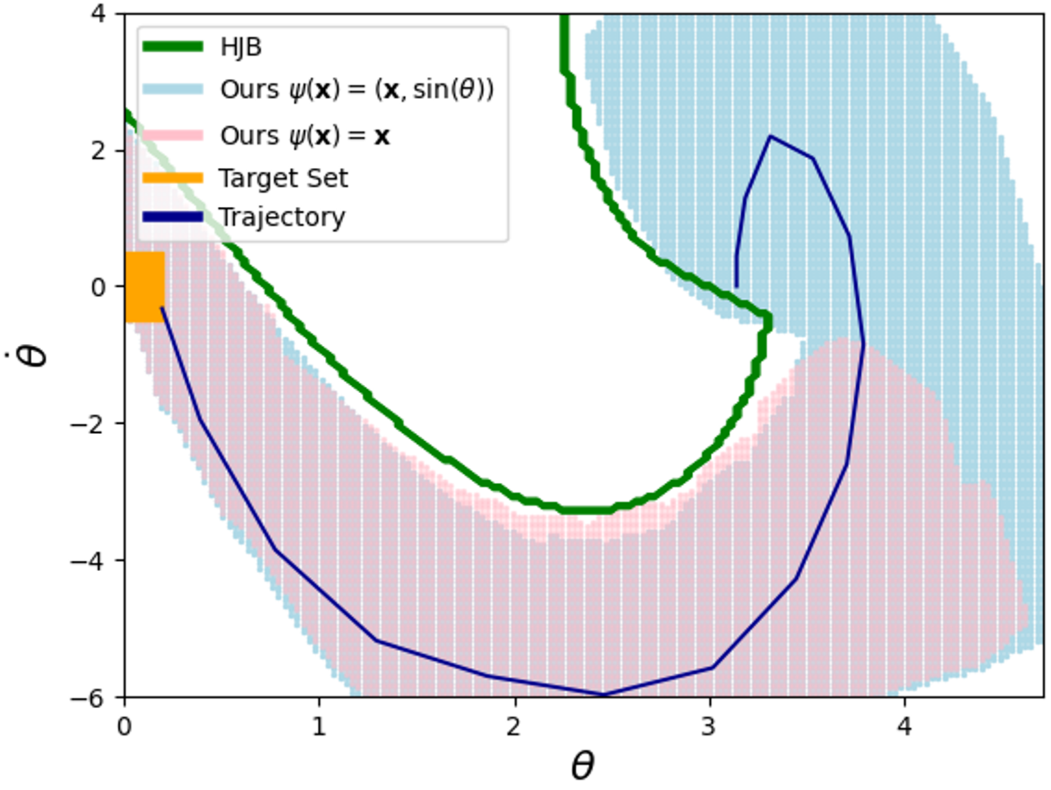
Local Koopman over-approximations

- Make local adaptations similar to hybridization
- Compute system matrices for each subdomain
  - Split the domain if the error bound is large
- The new error bound depends on:
  - How much the model changes
  - How much max training error decreases

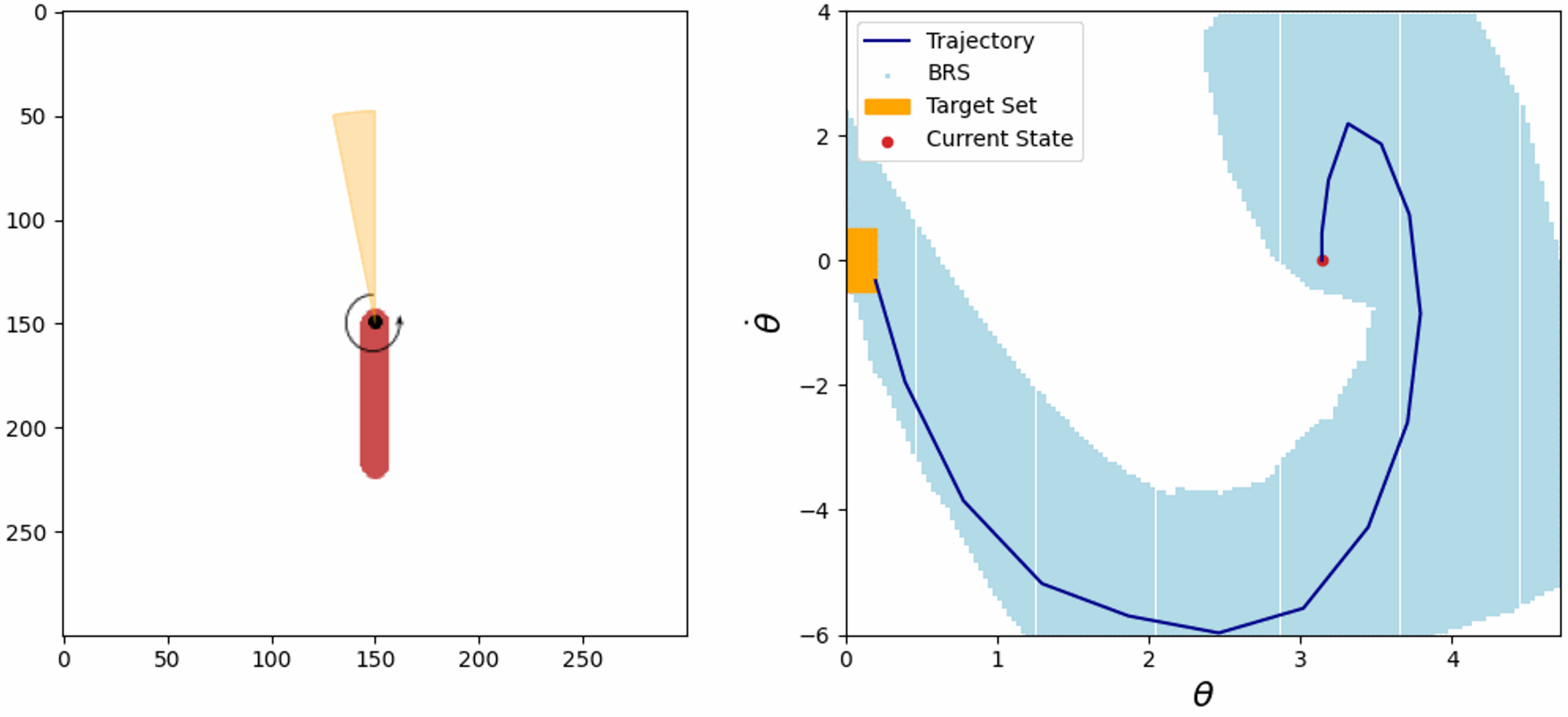


Inverted pendulum

	Num. of Sets	Comp. Time (s)
W/o lifting	640	952.6
With lifting	118	245.4



t = 0



0
50
100
150
200
250
0
50
100
150
200
250
Trajectory
BRS
Target Set
Current State
4
2
0
−2
−4
−6
0
1
2
3
4
$\dot{\theta}$
θ

Contribution

- A novel method to compute BRS inner-approximations for nonlinear systems
- Construction of data-driven KOA for BRS computation
- Online adaptation of KOA to localities that grant better approximations

Future works

- How to find the lifting function ψ ?
- Can we ensure having a lifting function improves BRS
- Replace polytopes by zonotopes

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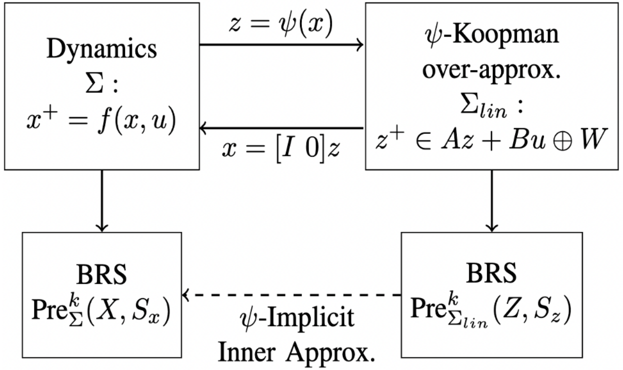

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