A Simulation Preorder for Koopman-like Lifted Control Systems

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Introduction

Finite-dimensional Koopman liftings have been successfully used to construct high dimensional linear approximations of dynamical systems, allowing to leverage linear control techniques to control nonlinear systems.

In this work, a simulation preorder among lifted systems — a generalization of finite-dimensional Koopman approximations to systems with inputs — is introduced. It is proved that this simulation relation implies the containment closed-loop behaviors.

These results enable us to compare different lifting functions and alternative lifted systems in terms of their usefulness in control design.

$$x^+ \in f_X(x,u) \;\underset{x = Cy}{\overset{y=\psi(x)}{\rightleftarrows}}\; \begin{cases} y^+ \in Ay + Bu \oplus W \\ x = Cy \end{cases}$$

Lifted systems

<u>Def.</u> A **lifted system** is defined as LS_Y: $\begin{cases} y(0) = \psi_Y(x(0)) \\ y(t+1) \in f_Y(y(t), u(t)) \\ x(t) = C_Y y(t) \end{cases}$

with $x(t) \in X$, $u(t) \in U$, $y(t) \in \mathbb{R}^{n_Y}$ and $C_Y \psi_Y(x) = x$.

<u>Important examples of lifted systems:</u>

1. **Unlifted** (i.e., classical) systems $x^+ \in f_X(x,u)$ are lifted systems with $n_Y = n_X$ and $\psi_Y = C_Y = id$.
2. **Affine** (or **piecewise affine**) lifted systems

$$y(t+1) \in Ay(t) + Bu(t) \oplus W$$
$$x(t) = Cy(t)$$

for which linear control methods can be used.

<u>Def.</u> The **behavior** of LS_Y under a policy π is the set

$$\boldsymbol{B}_\pi[LS_Y] = \{\, (\boldsymbol{x}, \boldsymbol{u}) \mid \exists \boldsymbol{y} \text{ s.t. } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}) \text{ is a max. sol.} \;\&\; u(t) = \pi(x(0), \dots, x(t)) \,\}.$$

<u>Def.</u> A **specification** is a set of (finite or infinite) sequences of (x,u) pairs: $S \subseteq (X \times U)^\infty$. *(e.g., safety or LTL constraints)*

<u>Def.</u> The specification S is **satisfied** by the lifted system LS_Y under the policy π if $\boldsymbol{B}_\pi[LS_Y] \subseteq S$. This is written $LS_Y \vDash_\pi S$.

Simulation between lifted systems

<u>Def.</u> LS_Y is **simulated** by LS_Z (denoted $LS_Y \preccurlyeq LS_Z$) if there exists a set-valued map $\rho: \mathbb{R}^{n_Z} \rightrightarrows \mathbb{R}^{n_Y}$ s.t.

- $\forall x \in X: \psi_Y(x) \in \rho(\psi_Z(x))$ *(Relation between liftings)*
- $\forall (z,u) \in \mathbb{R}^{n_Z} \times U: f_Y(\rho(z), u) \subseteq \rho(f_Z(z,u))$ *(dynamics)*
- $\forall z \in \mathbb{R}^{n_Z}: C_Y \rho(z) \subseteq \{C_Z z\}$ *(outputs)*

Theorem Given two lifted systems LS_Y and LS_Z and a policy π, if LS_Y is simulated by LS_Z, then the closed loop behavior of LS_Y under π is included in the closed loop behavior of LS_Z under π, i.e.,

$$LS_Y \preccurlyeq LS_Z \implies \boldsymbol{B}_\pi[LS_Y] \subseteq \boldsymbol{B}_\pi[LS_Z]$$

Consequently, if a specification S is satisfied by LS_Z under the policy π, then S is also satisfied by LS_Y under the same policy π, i.e.,

$$LS_Y \preccurlyeq LS_Z \vDash_\pi S \implies LS_Y \vDash_\pi S$$

If a nonlinear system of interest LS_X (e.g., unlifted) is simulated by an affine lifted system LS_Y, then linear control methods can be used to find a policy π s.t. $LS_Y \vDash_\pi S$. The policy can be used to control LS_X.

If LS_X is simulated by two lifted systems LS_Y and LS_Z and if $LS_Y \preccurlyeq LS_Z$, then LS_Y is a "not worse" representation of LS_X than LS_Z in terms of specification satisfaction.

<u>Some special cases of $LS_Y \preccurlyeq LS_Z$:</u>

If LS_Y is unlifted and

- LS_Z is affine
  → reduces to **Koopman over-approximation** in [1]
- LS_Z is affine and both systems are autonomous
  → reduces to **approximate immersion** in [2]
- LS_Z is picewise affine and unlifted
  → reduces to **hybridization** in [3]

Computational aspects

Given two lifted systems LS_Y and LS_Z, verifying if $LS_Y \preccurlyeq LS_Z$ is an **infinite dimensional** feasibility problem.

→ We derive **finite-dimensional** sufficient conditions to

1. find an affine lifted system that simulates a polynomial system
2. verify if one affine lifted system simulates another

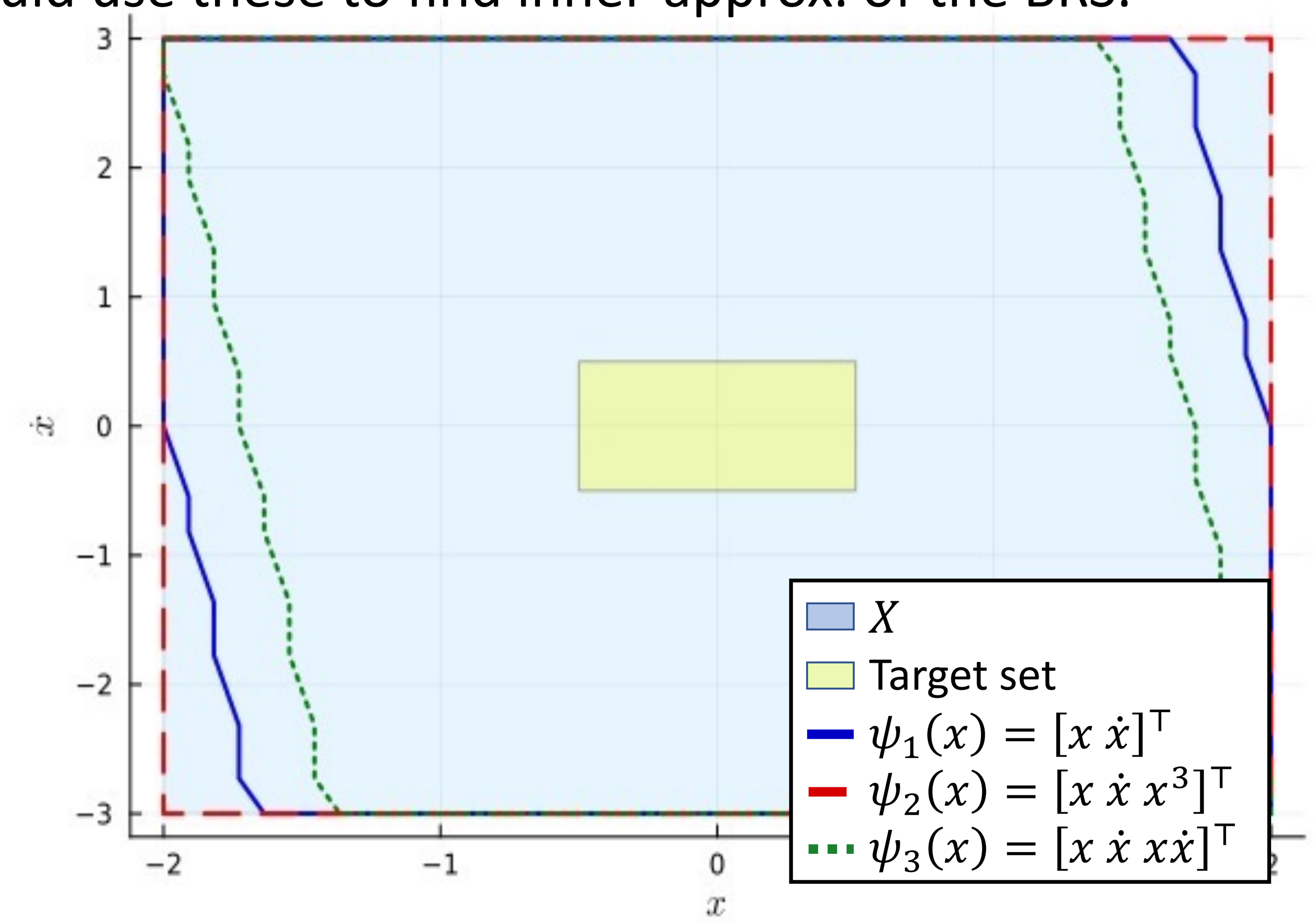
In practice…

Given a classical (i.e., unlifted) system LS_X: $x^+ \in f_X(x,u)$

1. Pick K lifting functions $\psi_1, \dots, \psi_K$
2. For each, compute an affine lifted system: LS_k s.t. $LS_X \preccurlyeq LS_k$
3. If $LS_i \preccurlyeq LS_j$, use LS_i instead of LS_j

Experiments with Backward reachable sets (BRS)

For the system LS_X: $\ddot{x} = 2x - 2x^3 - 0.5\dot{x} + u$ and three lifting functions, we find three affine lifted systems simulating LS_X. We could use these to find inner-approx. of the BRS.



Verifying $LS_Y \preccurlyeq LS_Z$ could be done in some (limited) cases.

Future works

Improve numerical methods to verify $LS_Y \preccurlyeq LS_Z$ and generalize to continuous-time and hybrid systems.

[1] Balim, H., Aspeel, A., Liu, Z., & Ozay, N. (2023). Koopman-inspired Implicit Backward Reachable Sets for Unknown Nonlinear Systems. *IEEE L-CSS*.
[2] Wang, Z., Jungers, R. M., & Ong, C. J. (2023). Computation of invariant sets via immersion for discrete-time nonlinear systems. *Automatica*.
[3] Girard, A., & Martin, S. (2011). Synthesis for constrained nonlinear systems using hybridization and robust controllers on simplices. *IEEE TAC*.

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