Koopman-inspired Implicit Backward Reachable Sets for Unknown Nonlinear Systems (L-CSS & CDC 2023)

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Introduction

Koopman liftings have been successfully used to learn high dimensional linear approximations for autonomous systems for prediction purposes, or for control systems for leveraging linear control techniques to control nonlinear dynamics.

In this work, we show how learned Koopman approximations can be used for correct-by-construction control. To this end, we introduce the Koopman over-approximation, a lifted representation that has a simulation-like relation with the underlying dynamics. Then, we prove how successive application of controlled predecessor operation in the lifted space leads to an implicit backward reachable set for the actual dynamics.

Backward Reachable Sets (BRS)

Set of points from which there exists a control sequence leading to a given target set.

Definition: Given

- an uncertain system $\Sigma: x^+ \in F(x,u),\ u \in U$
- a target set X
- a safe set S_x

the **1-step BRS** is defined as

$Pre_\Sigma(X, S_x) \coloneqq \{x \in S_x \mid \exists u \in U: F(x,u) \subseteq X\}$.

The **k-step BRS** is defined recursively as

$$Pre_\Sigma^k(X, S_x) \coloneqq Pre_\Sigma(Pre_\Sigma^{k-1}(X, S_x), S_x)$$

with $Pre_\Sigma^0(X, S_x) = X$.

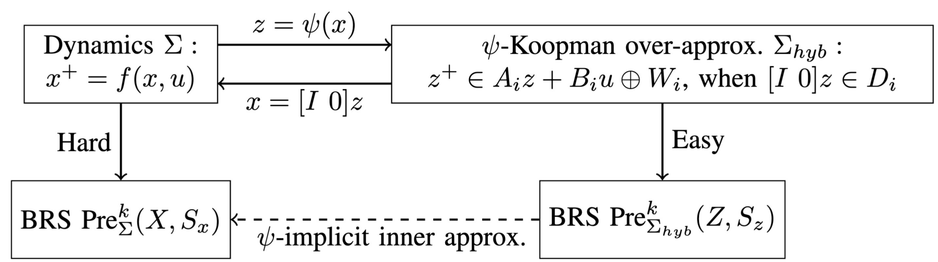
Goal

Computing BRS for nonlinear systems is intractable → Compute inner approximations.

Idea

1. BRS can be computed exactly for piecewise linear (a.k.a. hybrid) systems with polyhedral sets.
2. Koopman approach: Approximate nonlinear systems by (piecewise) linear ones in higher dimensions using a lifting function $\psi: x \mapsto \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$ s.t.

$$\psi(f(x,u)) \approx A_i \psi(x) + B_i u, \text{ when } x \in D_i.$$



Method (known dynamics)

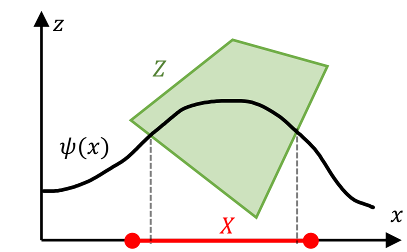
Definition: The piecewise linear system

$\Sigma_{hyb}: z^+ \in A_i z + B_i u \oplus W_i$, when $[I\ 0]z \in D_i$

is a **ψ-Koopman over approx.** of the system $\Sigma: x^+ = f(x,u)$ over S_x if for all $(x,u) \in S_x \times U$:

$\psi(f(x,u)) \in A_i \psi(x) + B_i u \oplus W_i$, when $x \in D_i$.

Definition: A set $Z \subseteq \mathbb{R}^{n_z}$ is a **ψ-implicit inner approx.** of a set $X \subseteq \mathbb{R}^{n_x}$ if $\{x \mid \psi(x) \in Z\} \subseteq X$.



Observation: If $\psi(x) = \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$, then for any set X, $Z \coloneqq \{z \mid [I\quad 0]z \in X\}$ is a ψ-implicit inner approx. of X.

Theorem: If

- Σ_{hyb} is a ψ-Koopman over approx. of Σ
- S_z is a ψ-implicit inner approx. of S_x
- Z is a ψ-implicit inner approx. of X,

then $Pre^k_{\Sigma_{hyb}}(Z, S_z)$ is a ψ-implicit inner approx. of $Pre^k_\Sigma(X, S_x)$ for all k.

Method (Unknown dynamics)

Goal: Given $\mathcal{D} = \{(x_i, u_i, x_i^+)\}_{i=1}^N$ with $x_i^+ = f(x_i, u_i)$ and ψ, compute a ψ-Koopman over approx. of the unknown f.

Definition: The **dispersion** of a data set $\mathcal{D}$ in S is $b_\mathcal{D} = \sup_{x \in S} \min_{x \in \mathcal{D}} \|x - x\|$.

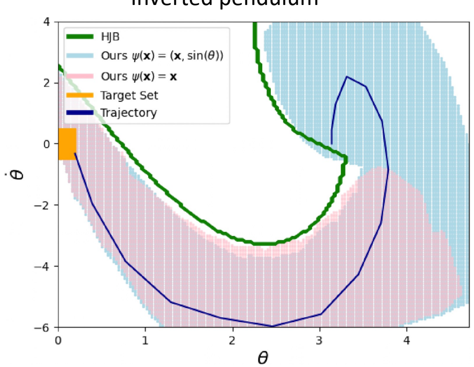
Lemma: If a data set $\mathcal{D}$ has dispersion $b_\mathcal{D}$ in S, then $\|f(x)\| \leq \max_{x \in \mathcal{D}} \|f(x)\| + Lip_f b_\mathcal{D}, \forall x \in S$.

Method (for one single piece):

1. Solve $e^* \coloneqq \min_{A,B,c} \|\overbrace{\psi(x^+) - A\psi(x) - Bu}^{E_{A,B}(x,u)} - c\|$.
2. Over-estimate $Lip_{E_{A^*,B^*}}$ using extreme value theory.
3. $\Sigma_{hyb}: z^+ \in A^* z + B^* u \oplus \mathcal{B}(c^*, e^* + b_\mathcal{D}\, Lip_{E_{A^*,B^*}})$ is a Koopman over approx.

Experiments

Inverted pendulum



Limitations

- How to choose the lifting function ψ?
- Increasing the lifting dimension can result in smaller BRS.

Future (& current) work

- Develop simulation-like relation between liftings: $\psi_1 \preccurlyeq \psi_2$.
- Use this simulation relation to construct monotonically better liftings.

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